Review Session:

1. Pricing Options based on binomial tree.

Given S_0, u, d, r and K.

For the payoff of certain option at maturity time N

$V_N = f(S_0, S_1, \ldots, S_N / K)$ VO then the values at maturity time are given as $V_N(\omega_1\omega_2 \cdots \omega_N) = f(S_0, S_1(\omega_1), S_2(\omega_1\omega_2), \ldots, S_N(\omega_1 \cdots \omega_N) / K)$ VO

For the previous time steps, $n = 0, 1, 2, \ldots, N-1$,

$$V_n(\omega_1\omega_2\cdots\omega_n) = \frac{1}{1+r}\left[\tilde{p}\, V_{n+1}(\omega_1\omega_2\cdots\omega_n H) + \tilde{q}\, V_{n+1}(\omega_1\omega_2\cdots\omega_n T)\right]$$

where $\tilde{p} = \frac{1+r-d}{u-d}$, $\tilde{q} = \frac{u-1-r}{u-d}$.

The number of shares in stock market if replicate the strategy of the option is

$$\Delta_n(\omega_1\omega_2\cdots\omega_n) = \frac{V_{n+1}(\omega_1\cdots\omega_n H) - V_{n+1}(\omega_1\cdots\omega_n T)}{S_{n+1}(\omega_1\cdots\omega_n H) - S_{n+1}(\omega_1\cdots\omega_n T)}, \quad 0 \le n \le N-1.$$

Ex: Consider Lookback option with payoff $V_3 = \max_{0\le n\le 3} S_n - S_3$
$S_0 = 100$, $u = 1.2$, $d = 0.8$, $r = 0.1$, $K = 100$.

① Compute the arbitrage-free price V_0

② Compute the number of stocks Δ_0 in the replicating strategy of the option at time zero.

2. Markov process and martingale.

Markov process: For every function $f(x)$, there is another function $g(x)$ such that $\tilde{E}_n[f(X_{n+1})] = g(X_n)$.

Basically, we need to find the relation between $f(x)$ and $g(x)$. (represent g by f).

Martingale: X_n only depends on first n coins and

$$X_n = \tilde{E}_n[X_{n+1}],$$

3. American Option:

If the payoff $V_N = f(S_0, S_1, \ldots, S_N | K) \vee 0$, then

$$V_N(\omega_1\omega_2\cdots\omega_N) = f(S_0, S_1(\omega_1), \ldots, S_N(\omega_1\cdots\omega_N)|K) \vee 0,$$

for previous steps

$$V_n(\omega_1\omega_2\cdots\omega_n) = \max\left\{ f(S_0, \cdots S_n(\omega_1\cdots\omega_n)|K), \frac{\tilde{E}_n[V_{n+1}](\omega_1\cdots\omega_n)}{1+r} \right\}.$$

The optimal stopping time

$$\tau^* = \min\{n;\ V_n = f(S_0, \ldots, S_n(\omega_1\cdots\omega_n)|K)\}.$$

We need to discuss optimal exercise policy for different sequences of events $\omega_1\omega_2\cdots\omega_n$.

4. Interest Rate Model.

The interest rates are given as

$$R_0,\ R_1(\omega_1),\ R_2(\omega_1\omega_2),\ \dots,\ R_N(\omega_1\cdots\omega_N).$$

Then discount process is given as

$$D_1 = \frac{1}{1+R_0},\quad \frac{D_2(\omega_1)}{D_1} = \frac{1}{1+R_1(\omega_1)},\ \dots,\ \frac{D_{n+1}(\omega_1\cdots\omega_n)}{D_n(\omega_1,\cdots,\omega_{n-1})} = \frac{1}{1+R_n(\omega_1\cdots\omega_n)}$$

The bond values at maturity time m

$$B_{n,m} = \frac{\tilde{E}_n[D_m]}{D_n}(\omega_1\cdots\omega_n).$$

The m-forward price at time n is

$$For_{n,m} = \frac{S_n}{B_{n,m}}.\qquad For_{m,m} = S_m.$$

and forward interest rate

$$F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1,$$

$$F_{m,m} = \frac{B_{m,m}}{B_{m,m+1}} - 1 = \frac{1}{\frac{1}{1+R_m}} - 1 = R_m,$$

(short an m-maturity zero-coupon bond get $B_{n,m}$, use the money buy $\frac{B_{n,m}}{B_{n,m+1}}$ share of $m+1$-maturity zero-coupon bond, pay 1 at time m and receive $\frac{B_{n,m}}{B_{n,m+1}}$ at time $m+1$)

The m-future price at time n is

$$Fut_{n,m} = \tilde{E}_n[S_m],\quad n=0,1,\dots,m.$$

$$Fut_{m,m} = S_m.$$